On the Propagation of Chaos for Recombination Models

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ABSTRACT

We consider binary interactions in an N-particle system. In particular, we use probability distributions known as recombination models to describe these interactions. Chaos propagates when the stochastic independence of two random particles in a particle system persists in time, as the number of particles tends to infinity. The concept of propagation of chaos was first introduced by Kac in connection with the Boltzmann equation, while modeling binary collisions in a gas. We obtain a development of Kac's program in the framework of recombination models. Specifically, our aim is to prove the relevant propagation of chaos phenomenon for our particle system. We first show that the solution for the master equation of our time-continuous process converges. Then, we use this solution together with the concepts of marginal measure and chaos to prove our desired result. Our main theorem for this study says that if a sequence of measures on our defined particle system is chaotic, then the resulting sequence of measures that had undergone the recombination process is also chaotic. This implies that the study of one particle after recombination gives information on the behavior of a group of particles in our particle system.

Keywords: particle system, recombination, master equation, propagation of chaos

INTRODUCTION

In 1956, Kac investigated the probabilistic foundations of kinetic theory and introduced the concept of propagation of chaos for the Boltzmann equation. Gottlieb (1998) stated in his thesis that Kac invented a class of interacting particle systems wherein particles collide at random with each other while the density of particles evolves deterministically in the limit of infinite particle number. He also described the concept of propagation of chaos in general, and stated that chaos propagates when the stochastic independence of two random particles in a particle system persists in time, as the number of particles tends to infinity.

Several topics in propagation of chaos play a major role in the field of probability, as discussed by Sznitman (1991). In particular, he wrote that one motivation for the subject was to investigate the connection between a detailed and a reduced description of the particles' evolution. He also stated that propagation of chaos deals with symmetric evolution of particles. That is, the study of one individual provides information on the behavior of the group. Another research related to this topic includes the paper of McKean (1967), where he introduced propagation of chaos for interacting diffusions and analyzed what are now called McKean-Vlasov equations. More recently in 2018, Thai studied a discrete version of the particle approximation of the McKean-Vlasov equations, and she proved the propagation of chaos property as well.

Many authors are concerned with proving that specific systems propagate chaos. In this study, we observe the propagation of chaos property for recombination models. Rabani et al. (1995) stated that random matings occur between parental chromosomes via a mechanism known as "crossover"; that is, children inherit pieces of genetic material from different parents according to some random rule. Recombination models are probability distributions that represent this randomness.

In 2016, Caputo and Sinclair stated that recombination models based on random mating have a number of applications in the natural sciences and play a significant role in the analysis of genetic algorithms. Moreover, they used quadratic dynamical systems to study the rate of convergence to equilibrium in terms of relative entropy for recombination models.

Let $\Omega_n = \{-1,1\}^n$ for a fixed natural number n. For each natural number N, we consider the state space $\Omega = \Omega_n^N$. We think of each state as a sequence of N particles in the system, where each particle is represented by a string of bits in Ω_n. Our main result in this paper is on the propagation of chaos phenomenon for recombination models that is represented by the interaction in this particle system.

MATERIALS AND METHODS

Recombination Models

Let $[n]$ denote the set $\{1, ..., n\}$. Given $A \subset [n]$ and $\sigma \in \Omega_n$, we write σ_A for the A-component of σ, that is, the subsequence $(\sigma_j, j \in A)$. For example, suppose $n = 5$ and let $\sigma = (-1, -1, 1, -1, 1)$, $A = \{2, 3, 5\}$. Then $\sigma_A = (-1, 1, 1)$.

Definition 1. (Recombination at a Set). *Given* $A \subset [n]$ *and* $(\sigma, \eta) \in \Omega_n \times \Omega_n$, *the recombination of* (σ, η) *at A consists in exchanging the A-component of* σ *with the A-component of* η. *This defines the map*

$$(\sigma, \eta) \to (\eta_A \sigma_{A^c}, \sigma_A \eta_{A^c}),$$

where A^c is the complement of set A, and the components of $\eta_A\sigma_{A^c} \in \Omega_n$ *is defined* by

$$(\eta_A\sigma_{A^c})_j = \begin{cases} \eta_j & \text{if } j \in A, \\ \sigma_j & \text{if } j \in A^C. \end{cases}$$

We have a similar definition for the components of $\sigma_A\eta_{A^c}$. For example, suppose n = 5 and let $\sigma = (-1,-1,1,-1,1), \eta = (-1,1,1,1,-1), A = \{2,3,5\}$. Then

$$\eta_A\sigma_{A^c} = (-1,1,1,-1,-1) \text{ and } \sigma_A\eta_{A^c} = (-1,-1,1,1,1).$$

The following process is motivated by the discussion of Carlen et al. (2010).

Definition 2. (Recombination Walk). *Let $S = (\sigma^{(1)}, \sigma^{(2)}, ..., \sigma^{(N)})$ denote a state in Ω, where $\sigma^{(i)} \in \Omega_n$ for each $i \in [N]$. S represents a set of N particles in our system. Recombination walk on Ω is a process that illustrates the evolution of an initial state S by applying recombinations on a random pair of particles at each step. We describe the process as follows:*

1. *Randomly pick a pair (k,l) of distinct indices in $[N]$, uniformly chosen from among all such pairs.*

2. *Choose a subset $A \subset [n]$ at random according to some probability distribution* ν.

3. *Update S by leaving $\sigma^{(i)}$ unchanged for $i \neq k,l$, and updating the particles $\sigma^{(k)}$ and $\sigma^{(l)}$ via the recombination at A.*

   Let $R_{k,l,A}S$ denote the new state in Ω. That is, if we assume $k < l$, the mapping $R_{k,l,A}$ on Ω is given by

$$(\sigma^{(1)}, \sigma^{(2)}, ..., \sigma^{(N)}) \to (\sigma^{(1)}, ..., \sigma^{(l)}_A\ \sigma^{(k)}_{A^c}, ..., \sigma^{(k)}_A, \sigma^{(l)}_{A^c}, \ldots, \sigma^{(N)}).$$

Here are some examples of recombination models represented by ν, from Caputo and Siclair (2016):

1. Single sire recombination: $v(A)\frac{1}{n}\sum_{1=1}^{n} 1(A=\{i\})$, where 1 is an indicator function on subsets of $[n]$. That is,

$$1(A=\{i\}) = \begin{cases} 1 \text{ if } A=\{i\} \text{ for some } i \in [n], \\ 0 \text{ otherwise.} \end{cases}$$

2. One-point crossover: $v(A) = \dfrac{1}{n+1}\sum_{i=0}^{n} 1(A=J_i)$, where $J_0 = \{\}$ *and* $J_i = [i]$ *for* $i \geq 1$.

3. Uniform crossover: $v(A)=\frac{1}{2^n}$, for all $A \subset [n]$.
4. The Bernoulli (q) model: for some q ∈ [0, 1/2],

$$v(A)=q^{|A|}(1-q)^{n-|A|}.$$

Master Equation

Let S_j denote the position after the j^{th} step of the walk and $\varphi \in R^{\Omega}$ be a bounded Borel-measurable function on Ω. We let $T:\varphi \to Q_N \varphi$ be the Markov transition operator on R^{Ω} represented by the transition matrix Q_N. Based on the defined process above, the function $(Q_N \varphi) \in R^{\Omega}$ is given by

$$\begin{aligned}(Q_N\varphi)(S) &= E[\varphi(S_{j+1}) \mid S_j = S] \\ &= \binom{N}{2}^{-1} \sum_{1\le k<l\le N} \sum_A v(A)\phi(R_{k,l,A}S).\end{aligned}$$

Let p_i be any probability measure on $\{-1, 1\}$ for $i \in [n]$ and let π_n be the direct product of the p_i's, that is, $\pi_n = p_1 \otimes p_2 \otimes \ldots \otimes p_n$. It is not difficult to see that the product measure $\pi = \pi_n^N$ on Ω gives us a reversible process, as defined by Levin et al. (2009):

Indeed, for any $S = (\sigma^{(1)}, \sigma^{(2)}, \ldots, \sigma^{(N)})$, we have

$$\pi(R_{k,l,A}S)=\left[\prod_{j\neq k,l}\pi_n(\sigma^{(j)})\right]\pi_n(\sigma_A^{(l)}\sigma_{A^c}^{(k)})\pi_n(\sigma_A^{(k)}\sigma_{A^c}^{(l)})=\pi(S).$$

Since $R_{k,l,A}(R_{k,l,A}S) = S$, the transition matrix Q_N is symmetric, and so we have

$$\pi(S)Q_N(S, R_{k,l,A}S) = \pi(R_{k,l,A}S)Q_N(R_{k,l,A}S, S),$$

where $Q_N(x, y) = P[S_{j+1} = y|S_j = x]$.

The following definition is adopted from Carlen, Degond and Wennberg (2013).

Definition 3. (Time-continuous Master Equation). The time-continuous masterequation associated to a discrete Markov process with transition matrix Q is given by

$$\frac{d}{dt}F^{(N)}(t,S)=L^*F^{(N)}(t,S) \text{ with } F^{(N)}(0,S)=F_0^{(N)}(S),$$

where $L^* = N[Q^* - I]$ with Q^* the adjoint of Q, I is the identity matrix, and $F_0^{(N)}$ is the initial probability density function.

Since the Q_N is symmetric, we have $Q^*_N = Q_N$. We now give our time-continuous master equation in the following proposition, which is motivated by Carlen et al. (2010).

Proposition 4. *If the law $\mu_0^{(N)}$ of the initial state S_0 has a density $F_0^{(N)}$ with respect to π, then for all $t > 0$, the law $\mu_t^{(N)}$ of S_t has a density $F_t^{(N)}$ with respect to π, where* $F_t^{(N)}$ is the solution of the following equation

$$\frac{d}{dt}F^{(N)} = L_N\mu^{(N)}\left(t,S\right) \text{ with } F^{(N)}(0,S) = F_0^{(N)}\left(S\right), \tag{1}$$

where $L_N = N(Q_N - I)$ and I is the identity matrix.

Remarks:

1. Note that L_N is self-adjoint, since Q_N is symmetric.

2. As defined in the proposition, we have $\mu_t^{(N)}(S) = F^{(N)}(t, S)\pi(S)$ and thus (1) is equivalent to

$$\frac{d}{dt}\mu^{(N)} = L_N\mu^{(N)}\left(t,S\right) \text{ with } \mu^{(N)}(0,S) = \mu_0^{(N)}\left(S\right). \tag{2}$$

3. The solution $\mu_t^{(N)}$ of (2) is given by

$$\mu_t^{(N)} = e^{tL_N}\mu_0^{(N)} = \sum_{j=0}^{\infty}\frac{t^j}{j!}L_N^j\mu_0^{(N)}.$$

Propagation of Chaos

We now present some known definitions needed for our theorem. The following definition is taken from Carlen et al. (2010).

Definition 5. (Marginal Measure). *Let $\mu^{(N)}$ be a probability measure on Ω and let k be a positive integer with $k < N$. The marginal measure of $\mu^{(N)}$ for the first k particles on $A \subset \Omega^k_n$ is defined as*

$$P_k(\mu^{(N)})[A] = \mu^{(N)}[\{(\sigma^{(1)}, \ldots, \sigma^{(k)}) \in A\}].$$

We adopt the following definition from Sznitman (1991).

Definition 6. (Chaos). *Let μ be a probability measure on Ω_n. Let $\{\mu^{(N)}\}_{N=1}^{\infty}$ be a sequence of symmetric probability measures on Ω, i.e., the value is the same no matter the order of its arguments. We say that $\{\mu^{(N)}\}_{N=1}^{\infty}$ is μ-chaotic if for $\{g_k : 1 \leqslant k < N\} \subset C_b(\Omega_n)$, where $C_b(\Omega_n)$ is the set of bounded functions from Ω to* R, *we have*

$$\lim_{N\to\infty}\sum_{S\in\Omega} g(S)\mu^{(N)}(S) = \prod_{t=1}^{k}\left(\sum_{\sigma\in\Omega_n} g_t(\sigma)\mu(\sigma).\right), \tag{3}$$

where $g = g_1 \otimes g_2 \otimes \ldots \otimes g_k \otimes \mathbf{1} \otimes \ldots \otimes \mathbf{1}$ with $N-k$ copies of the ***1***.

We adopt the following definition from Gottlieb (1998).

Definition 7. (Propagation of Chaos). *A sequence $\{Q_N\}_{N=1}^{\infty}$ whose N-th term is a Markov transition function on Ω propagates chaos if, whenever a sequence of measures $\{\mu^{(N)}\}_{N=1}^{\infty} \subset \Omega$ is μ-chaotic for some measure $\mu \in \Omega_n$, then for any $t \geq 0$, the sequence $\{\mu_t^{(N)}\}_{N=1}^{\infty} \subset \Omega$ which satisfies (2) is $\bar{\mu}_t$-chaotic for some measure $\bar{\mu}_t \in \Omega_n$.*

RESULTS AND DISCUSSION

We now present our main result for this study in the following theorem.

Statement of the Main Result

Theorem 8. *(Propagation of Chaos for Recombination Models).*

Let $\mu^{(N)}$ be the probability measure on Ω with probability density $F^{(N)}$ with respect to π. Suppose that $\{\mu^{(N)}\}_{N=1}^{\infty}$ is a μ-chaotic family, where μ is some probability measure on Ω_n. For each natural number N, let $F^{(N)}(t,\cdot)$ denote the solution of (1) at time t, starting from the initial data $F^{(N)}$. Let $\mu^{(N)}(t,\cdot)$ be the measure on Ω with probability density $F^{(N)}(t,\cdot)$, with initial measure $\mu^{(N)}$ at $t = 0$. Then for any $t \geq 0$, $\{\mu^{(N)}(t,\cdot)\}_{N=1}^{\infty}$ is $\mu(t,\sigma)$-chaotic, where $\mu(t,\sigma)$ is the solution of the following problem:

$$\begin{cases} \mu(0,\cdot) = \mu \\ \dfrac{d}{dt}\mu(t,\sigma) = \displaystyle\sum_A v(A)[\mu_A(t,\sigma_A)\otimes\mu_{A^c}(t,\sigma_{A^c}) - \mu(t,\sigma)], \end{cases} \tag{4}$$

where

$$\mu_A(t,\sigma_A) = \sum_{\sigma A^c}\mu(t,\sigma).$$

Remarks:

1. For each natural number N, we will use the notation $\mu^{(N)}(\cdot)$ instead of $\mu^{(N)}(0, \cdot)$.

2. From the discussion of Hauray and Mischler (2014): Suppose $\mu^{(N)}$ is symmetric. Define the empirical measure of the system as

$$M_n(t,\Sigma) = \frac{1}{N}\sum_{t=1}^{N}\delta_{\sigma^{(1)}(t)}(\Sigma),$$

where $\sum$ is a subset of Ω_n *and* δ is the Dirac measure. That is,

$$\delta_{\sigma^{(1)}(t)}(\Sigma) = \begin{cases} 1, \text{ if } \sigma^{(i)}(t) \in \Sigma, \\ 0, \text{ otherwise.} \end{cases}$$

Then $\mu^{(N)}$ is μ-chaotic is equivalent to saying that

$$\lim_{N\to\infty} M_N(t,\Sigma) \;=\; \mu(t,\Sigma).$$

Moreover, it is also equivalent to condition (3) with $k = 2$. We will use this later in the proof of our theorem.

3. The differential equation given in (4) has a unique solution, as shown by Baake et al. (2016) in the general case. In our theorem, this solution is given by $\mu(t, \sigma)$, which we will choose appropriately in our proof.

We will first prove three lemmas that will be needed for the proof of our theorem.

Three Lemmas

The first lemma that we will prove concerns the convergence of the series that we will get from the solution of (2).

Lemma 9. *Let $g \in C_b(\Omega)$ that depends only on $\sigma^{(1)},\ldots,\sigma^{(k)}$. If $T < 1/4$, for any natural number N, the power series*

$$e^{tL_N} g = \sum_{j=0}^{\infty}\frac{t^j}{j!}L_N^j g, \tag{5}$$

absolutely converges for $t \in [0, T]$.

Proof: Suppose that g depends only on one particle and, without loss of generality, we can set this to be the first particle. That is, for some $\bar{g} : \Omega n \to R$, we have

$$g(S) = g(\sigma^{(1)}, \ldots, \sigma^{(N)}) = \bar{g}(\sigma^{(1)}).$$

Since $g \in C_b(\Omega)$, this implies that $\bar{g} \in C_b(\Omega_n)$, that is for all $\sigma \in \Omega_n$,

$$|\bar{g}(\sigma)| < M \text{ for some } M \in R.$$

Let N be a natural number. Since $L_N = N(Q_N - I)$, by using the definition of Q_N, we have for any $S = (\sigma^{(1)}, \sigma^{(2)}, \ldots, \sigma^{(N)})$ that

$$\begin{aligned} L_N g(S) &= N \binom{N}{2}^{-1} \sum_{1 \le k < l \le N} \sum_A v(A)[g(R_{k,l,A}S) - g(s)] \\ &= \frac{2}{N-1} \sum_{l=2}^{N} \sum_A v(A)[\bar{g}(\sigma_A^{(l)} \sigma_{A^c}^{(1)}) - \bar{g}(\sigma^{(1)})] \end{aligned}$$

So that

$$\left| L_N g(S) \right| < \frac{2}{N-1} \sum_{l=2}^{N} \sum_A \left| v(A) \right| (2M) = 4M. \tag{6}$$

Define g_2 by

$$g_2(\sigma^{(1)}, \sigma^{(l)}) = \sum_A v(A)[\bar{g}(\sigma_A^{(l)} \sigma_{A^c}^{(1)}) - \bar{g}(\sigma^{(1)})], \text{ for } l \in \{2, \ldots, N\}.$$

Then

$$L_N g(S) = \frac{2}{N-1} \sum_{l=2}^{N} g_2(\sigma^{(1)}, \sigma^{(l)}).$$

In general, we will prove by induction that

$$\left| L_N^j g(S) \right| < j! 4^j M, \tag{7}$$

for any natural number j.

Indeed j = 1 is true by (6). Suppose $\left|L_N^j g(S)\right| < j!4^j M$ for some natural number k. Let

$$\bar{g}_{k+1} = L_N^k g, \ \text{ where } \bar{g}_{k+1} : \Omega_n^{k+1} \to \mathbb{R}.$$

Then $$L_N^{k+1} g(S) = L_N \bar{g}_{k+1}(S^{(k+1)})$$

$$= \frac{2}{N-1} \sum_A v(A) \sum_{1 \le l_1 < l_2 \le k+1} [\bar{g}(R_{l_1,l_2}, AS^{(k+1)}) - \bar{g}(S^{(k+1)})]$$

$$+ \frac{2}{N-1} \sum_{l=1}^{k+1} \sum_{j=k+2}^{N} \sum_A v(A)[\bar{g}(\sigma^{(1)},...,\sigma_A^{(j)}\sigma_{A^c}^{(l)},...,\sigma^{(k+1)}) - \bar{g}(S^{(k+1)})]$$

So that

$$\begin{aligned}
&\left|L_N^{k+1} g(S)\right| \\
&< \frac{2}{N-1}\binom{k+1}{2} 2\left|\bar{g}(S^{(k+1)})\right| + \frac{2}{N-1}(k+1)(N-k-1)2\left|\bar{g}(S^{(k+1)})\right| \\
&= 4\left|\bar{g}(S^{(k+1)})\right|(k+1)(\frac{2N-K-2}{2N-2}) \\
&< 4(k!4^k M)(k+1) = (k+1)!4^{k+1} M
\end{aligned}$$

Now, for $0 \le t < 1/4$, the series for $e^{tL_N}g$ absolutely converges since

$$\sum_{j=0}^{\infty} \frac{t^j}{j!}\left|L_N^j g\right| < \sum_{j=0}^{\infty} \frac{t^j}{j!} j!4^j M = M\sum_{j=0}^{\infty} (4t)^j.$$

In general, if we start with g, a functional of p particles where $p \ge 2$, (7) is changed into

$$|L^j{}_N g(S)| < (j + p - 1)!4^j M. \tag{8}$$

for all natural numbers $N, S \in \Omega$.

Furthermore, for any natural number j,

$$\left(j+p-1\right)!4^j M = j!\frac{\left(j+p-1\right)!}{j!}\, 4^j M < j!\left(j+p-1\right)^{p-1} 4^j M.$$

So that for $0 \le t < 1/4$ and any natural number N, we have

$$\sum_{j=0}^{\infty} \frac{t^j}{j!}\left|L_N^j g\right| < \sum_{j=0}^{\infty} (4t)^j (j+p-1)^{p-1} M.$$

Thus, $e^{tL_N}g$ is absolutely convergent for $0 \leq t < 1/4$. This ends the proof of the lemma.

Before we proceed to the second lemma, we first look at the following observation.

For each $k \in [N-1]$, we have

$$P_k(\mu^{(N)}(t,S^{(k)})) = \sum_{\sigma^{(k+1)},\dots,\sigma^{(N)}} \mu^{(N)}(t,S).$$

Let g be a function that depends on $\sigma^{(1)}$ and define g_2 as we have defined before.

Then

$$\sum_S L_N g(S)\mu^{(N)}(S) = \sum_{\sigma^{(1)},\sigma^{(2)}} \left[\sum_{\sigma^{(3)},\dots\sigma^{(N)}} \frac{2}{N-1} \sum_{l=2}^{N} g_2(\sigma^{(1)},\sigma^{(l)})\mu^{(N)}(S) \right].$$

So that,

$$\lim_{N\to\infty} \sum_{S\in\Omega} L_N g(S)\mu^{(N)}(S) = 2\left[\sum_{S^{(2)}\in\Omega^{(2)}} \mu_2(S^{(2)})g_2(S^{(2)}) \right],$$

where

$$\mu_2(\sigma^{(1)},\sigma^{(2)}) = \lim_{N\to\infty} P_2(\mu^{(N)}(0,\sigma^{(1)},\sigma^{(2)})).$$

We now prove a more general result in the following lemma.

Lemma 10. *Let g be a function of one variable, say* $\sigma^{(1)}$. *Then for any natrual number k,*

$$\lim_{N\to\infty} \sum_{S\in\Omega} L_N^k g(S)\mu^{(N)}(S) = 2^k \left[\sum_{S^{(k+1)}\in\Omega^{(k+1)}} \mu_{k+1}(S^{(k+1)})g_{k+1}(S^{(k+1)}) \right], \tag{9}$$

where $\mu_{k+1}(S^{(k+1)}) = \lim_{N\to\infty} P_{k+1}(\mu^{(N)}(0,S^{(k+1)})), g_1 = g,$ and

$$g_{k+1}(S^{(k+1)}) = \sum_{t=1}^{k}\sum_A v(A)\left[g_k(\sigma^{(1)},\dots,\sigma^{(t-1)},\sigma_A^{(k+1)}\sigma_{A^c}^{(t)},\dots,\sigma^{(k)}) - g_k(S^{(k)}) \right]. \tag{10}$$

Proof: Note that for $k \geq 2$ we have

$$L_N^k g(S) = \frac{2^k}{N-1} \sum_A v(A) \sum_{1 \le l_1 < l_2 \le k} [g_{k-1}(R_{l_1, l_2, A} S^{(k-1)}) - g_{k-1}(S^{(k-1)})]$$

$$+ \frac{2^k}{N-1} \sum_{J=k+1}^{N} g_{k+1}(\sigma^{(1)}, \ldots, \sigma^{(k)}, \sigma^{(j)})$$

$$= \alpha(S) + \beta(S),$$

where α(S) and β(S) represent the 1st and 2nd set of summations in the right-hand side,respectively. Since the summations in α(S) involve a finite number of terms, we can write it as $\alpha(S) = \frac{C}{N-1}$ for some constant C. So that,

$$\lim_{N \to \infty} \sum_{S \in \Omega} \alpha(S) \mu^{(N)}(S) = \lim_{N \to \infty} \frac{C}{N-1} \sum_{S \in \Omega} \mu^{(N)}(S) = \lim_{N \to \infty} \frac{C}{N-1} = 0.$$

Now, by symmetry we have

$$\sum_{S \in \Omega} \beta(S) \mu^{(N)}(S) = \sum_{\sigma^{(1)}, \ldots, \sigma^{(k+1)}} \left[\sum_{\sigma^{(k+2)}, \ldots, \sigma^{(N)}} \frac{2^k (N-k)}{N-1} g_{k+1}(S^{(k+1)}) \mu^{(N)}(S) \right].$$

So that

$$\lim_{N \to \infty} \sum_{S \in \Omega} \beta(S) \mu^{(N)}(S) = 2^k \left[\sum_{S^{(k+1)} \in \Omega^{(k+1)}} \mu_{k+1}(S^{(k+1)}) g_{k+1}(S^{(k+1)}) \right].$$

This ends the proof of the lemma.

Finally, we prove the third lemma that will be essential to the proof of our theorem.

Lemma 11. *Let g_1 and h_1 be functions of a single variable. Let γ_2 be a function of two variables, say $\sigma^{(1)}$ and $\sigma^{(2)}$, such that $\gamma_2(\sigma^{(1)}, \sigma^{(2)}) = g_1(\sigma^{(1)}) h_1(\sigma^{(2)})$. Then for* $0 \le t < 1/4$,

$$\sum_{l=0}^{\infty}\frac{(2t)^l}{l!}\left[\sum_{S^{(l+2)}}\gamma_{l+2}(S^{(l+2)})\right]\prod_{t=1}^{l+2}\mu_1(\sigma^{(t)})$$

$$=\sum_{j=0}^{\infty}\frac{(2t)^j}{j!}\left[\sum_{S^{(j+1)}}g_{j+1}(S^{(j+1)})\right]\prod_{t=1}^{j+1}\mu_1(\sigma^{(t)}) \tag{11}$$

$$\times\sum_{k=0}^{\infty}\frac{(2t)^k}{k!}\left[\sum_{S(k+1)}h_{k+1}(S^{(k+1)})\right]\prod_{t=1}^{k+1}\mu_1(\sigma^{(t)}),$$

where $g_{j+1}, h_{k+1}, \gamma_{l+2}$ *are defined inductively as in* (10) *for* $j, k, l \geq 0$.

Proof: Let *RHS* demote the hand side of equation (11). Distributing the terms in the *RHS* gives us.

$$RHS=\sum_{l=0}^{\infty}\frac{(2t)^l}{l!}\sum_{S^{(l+2)}}\sum_{m=0}^{l}\binom{l}{m}\left[g_{m+1}(\sigma^{(1)},\sigma^{(3)},\ldots,\sigma^{(m+2)})\right]$$

$$\times\left[h_{(l-m)+1}(\sigma^{(2)},\sigma^{(m+3)},\ldots,\sigma^{(l+2)})\right]\prod_{t=1}^{l+2}\mu_1(\sigma^{(t)}).$$

The result is proved if we show that for each $l \geq 0$,

$$\sum_{S^{(l+2)}}\gamma_{l+2}(S^{(l+2)})G(S^{(l+2)})$$

$$=\sum_{S^{(l+2)}}\sum_{m=0}^{l}\binom{l}{m}\left[g_{m+1}(\sigma^{(1)},\sigma^{(3)},\ldots,\sigma^{(m+2)})\right]$$

$$\times\left[h_{(l-m)+1}(\sigma^{(2)},\sigma^{(m+3)},\ldots,\sigma^{(l+2)})\right]G(S^{(l+2)}), \tag{12}$$

for any symmetric function *G*.

The case $l = 0$ is true by the definition of γ_2, that is

$$\sum_{S^{(2)}}g_1(\sigma^{(1)})h_1(\sigma^{(2)})G(S^{(2)})=\sum_{S^{(2)}}\gamma_2(S^{(2)})G(S^{(2)}).$$

The proof readily follows by using induction on this initial condition.

Proof of the Theorem

We now proceed with the proof of our theorem.

Proof Theorem 4: Again, we consider the case when $g \in C_b(\Omega)$ that depends only on one variable, say $\sigma^{(1)}$. Since L_N is self-adjoint, we have

$$\sum_{S\in\Omega} g(S)\mu^{(N)}(t,S) = \sum_{S\in\Omega} g(S)e^{tL_N}\mu^{(N)}(S) = \sum_{S\in\Omega}\sum_{k=0}^{\infty}\frac{t^k}{k!}L_N^k g(S)\mu^{(N)}(S).$$

Since we are assuming that $\{\mu(N)\}_{N\in\mathbb{N}}$ is μ- chaotic, we have from (9)

$$\lim_{N\to\infty}\sum_{S\in\Omega} L^k g(S)\mu^{(N)}(S) = 2^k\left[\sum_{S(k+1)\in\Omega(k+1)} g_{k+1}(S^{(k+1)})\right]\prod_{t=1}^{k+1}\mu_1(\sigma^{(t)}),$$

where $\mu_1(0) = \lim_{N\to\infty} P_1(\mu^{(N)}(0,\sigma))$.

Note also that

$$\sum_{S\in\Omega} g(S)\mu^{(N)}(t,S) = \sum_{\sigma^{(1)}} g(\sigma^{(1)})\left[\sum_{\sigma^{(2)},\dots,\sigma^{(N)}}\mu^{(N)}(t,S)\right]$$

$$= \sum_{\sigma^{(1)}} g(\sigma^{(1)})P1(\mu^{(N)}(t,\sigma^{(1)})).$$

From (5), it follows that for $0 \leq t < 1/4$,

$$\sum_{\sigma^{(1)}} g(\sigma^{(1)})\bar{\mu}_1(t,\sigma^{(1)}) = \sum_{k=0}^{\infty}\frac{(2t)^k}{k!}\left[\sum_{s^{(k+1)}\in\Omega^{(k+1)}} g_{k+1}(S^{(k+1)})\right]\prod_{t=1}^{k+1}\mu_1(\sigma^{(t)}), \tag{13}$$

where $\bar{\mu}(t,\ \sigma^{(1)}) = \lim_{N\to\infty} P_1(\mu^{(N)}(t,\sigma^{(1)}))$.

Starting now from a function $\gamma_2(\sigma^{(1)}, \sigma^{(2)}) = g(\sigma^{(1)})h(\sigma^{(2)})$ are defining $\gamma_k(S^{(k)})$ inductively as in (10), we obtain again for $0 \leq t < 1/4$,

$$\sum_{\sigma^{(1)},\sigma^{(2)}} g(\sigma^{(1)})h(\sigma^{(2)})\mu_2(t,\sigma^{(1)},\sigma^{(2)})$$

$$= \sum_{k=0}^{\infty}\frac{(2t)^k}{k!}\left[\sum_{S^{(k+2)}\in\Omega^{(k+2)}}\gamma_{k+2}(S^{(k+2)})\right]\prod_{t=1}^{k+2}\mu_1(\sigma^{(t)}),$$

where $\mu_2(t,\ \sigma^{(1)},\sigma^{(2)})\ =\ \lim_{N\to\infty} P_2(\mu^{(N)}(t,\sigma^{(1)},\sigma^{(2)}))$.

From (11) and (13) it follows that

$$\sum_{\sigma^{(1)},\sigma^{(2)}} g(\sigma^{(1)})h(\sigma^{(2)})\mu_2(t,\sigma^{(1)},\sigma^{(2)})$$

$$=\left[\sum_{\sigma^{(1)}} g(\sigma^{(1)})\bar{\mu}_1(t,\sigma^{(1)})\right]\left[\sum_{\sigma^{(1)}} h(\sigma^{(2)})\bar{\mu}_1(t,\sigma^{(2)})\right]. \tag{14}$$

Since g, h are arbitrarily chosen, we have $0 \le t < 1/4$,

$$\mu_2(t,\sigma^{(1)},\sigma^{(2)}) = \bar{\mu}_1(t,\sigma^{(1)})\bar{\mu}_1(t,\sigma^{(2)}).$$

We only need to show that $\bar{\mu}_1(t,\sigma^{(1)})$ is the solution of (4). Similar computations will lead to

$$\sum_{\sigma^{(1)},\sigma^{(2)}} g_2(\sigma^{(1)},\sigma^{(2)})\bar{\mu}_1(t,\sigma^{(1)}),\bar{\mu}_2(t,\sigma^{(2)})$$

$$=\sum_{k=0}^{\infty}\frac{(2t)^k}{k!}\left[\sum_{S^{(k+2)}\in\Omega^{(k+2)}} g_{k+2}(S^{(k+2)})\right]\prod_{t=1}^{k+2}\mu_1(\sigma^{(t)}). \tag{15}$$

Now differentiating (13), we get

$$\sum_{\sigma^{(1)}} g(\sigma^{(1)})\frac{d}{d_t}\bar{\mu}_1(t,\sigma^{(1)})$$

$$=\sum_{k=1}^{\infty}\frac{(2t)^{k-1}}{(k-1)!}\left[\sum_{S^{(k+1)}\in\Omega^{(k+1)}} g_{k+1}(S^{(k+1)})\right]\prod_{t=1}^{k+1}\mu_1(\sigma^{(t)}).$$

Using a change of variable for the summation in the right hand side,

$$\sum_{\sigma^{(1)}} g(\sigma^{(1)}) \frac{d}{d_t} \bar{\mu}_1(t,\sigma^{(1)})$$

$$= \sum_{k=0}^{\infty} \frac{(2t)^k}{k!} \left[\sum_{S^{(k+2)} \in \Omega^{(k+2)}} g_{k+2}(S^{(k+2)}) \right] \prod_{t=1}^{k+2} \mu_1(\sigma^{(t)})$$

$$= \sum_{\sigma^{(1)},\sigma^{(2)}} g_2(\sigma^{(1)},\sigma^{(2)}) \bar{\mu}_1(t,\sigma^{(1)}) \bar{\mu}_1(t,\sigma^{(2)})$$

$$= \sum_{\sigma^{(1)},\sigma^{(2)}} (Q_2 - I) g(\sigma^{(1)}) \bar{\mu}_1(t,\sigma^{(1)}) \bar{\mu}_1(t,\sigma^{(2)}).$$

Again, by using the self-adjoint property, we have

$$\sum_{\sigma^{(1)}} g(\sigma^{(1)}) \frac{d}{d_t} \bar{\mu}_1(t,\sigma^{(1)})$$

$$= \sum_{\sigma^{(1)},\sigma^{(2)}} g(\sigma^{(1)}) (Q_2 - I) \bar{\mu}_1(t,\sigma^{(1)}) \bar{\mu}_1(t,\sigma^{(2)})$$

$$= \sum_{\sigma^{(1)}} \sum_{\sigma^{(2)}} g(\sigma^{(1)}) \left[\sum_{A} v(A) [\bar{\mu}_1(t,\sigma_A^{(2)}\sigma_{A^c}^{(1)} - \bar{\mu}_1(t,\sigma^{(1)})] \right] \bar{\mu}(t,\sigma^{(2)})$$

$$= \sum_{\sigma^{(1)}} g(\sigma^{(1)}) \sum_{A} v(A) \left[\sum_{\sigma^{(2)}} \left(\bar{\mu}_1(t,\sigma_A^{(2)}\sigma_{A^c}^{(1)})(\bar{\mu}_1(t,\sigma^{(2)}) \right) - \bar{\mu}(t,\sigma^{(1)}) \right]$$

$$= \sum_{\sigma^{(1)}} g(\sigma^{(1)}) \sum_{A} v(A) [\bar{\mu}_{1A}(t,\sigma_A^{(1)}) \otimes \bar{\mu}_{1A^c}(t,\sigma_{A^c}^{(1)}) - \bar{\mu}_1(t,\sigma^{(1)})]$$

Finally, we need to remove the restriction on $t < 1/4$. We can start with some t_0, where $0 \leq t_0 < 1/4$, and we repeat the argument to extend the proof to time t where $t_0 \leq t < t_0 + 1/4$. We can see that by proceeding in this manner, we can cover the whole time range, that is, $\mu^{(N)}(t,\cdot)$ is $\mu(t,\sigma)$-chaotic for any natural number N and $t \geq 0$. This ends the proof of the theorem.

CONCLUSION

In this study, we introduced some definitions and ideas used to prove propagation of chaos for a particle system where the binary interactions are represented by recombination models. Specifically, we showed that if a sequence of measures $\{\mu^{(N)}\}_{N=1}^{\infty}$ on our defined particle system Ω is chaotic, then for any $t \geq 0$, $\{\mu_t^{(N)}\}_{N=1}^{\infty}$, is also chaotic, where each measure $\mu_t^{(N)}$ is obtained after applying recombinations on $\mu^{(N)}$ for some time t.

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